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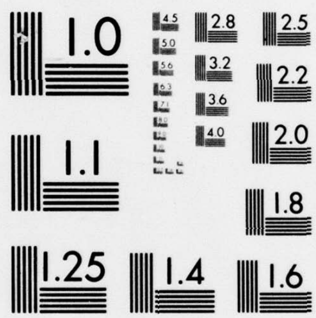
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POSITIVE KERNELS AND STOCHASTIC INTEGRALS

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ABSTRACT

This report is intended to expand the applicability of positive kernel theory to probabilistic settings and stochastic integrals. The main result states that if $a(t)$ is a positive kernel, and $\{\beta(t) : t \geq 0\}$ a Brownian motion, then

$$(*) \qquad \int_0^T \xi(t) \int_0^t a(t-\tau)\xi(\tau)d\beta(\tau)d\beta(t) + \frac{1}{2}a(0)\int_0^T |\xi(t)|^2 dt \geq 0, \quad \text{a.s.}$$

for every stochastic process $\{\xi(t) : t \geq 0\}$ which has a stochastic differential $d\xi(t)$ with respect to $\beta(t)$, and for every $T \geq 0$. The implication of (*) concerning energy estimates for certain Ito-Volterra equations is discussed, and examples are provided.

AMS (MOS) Subject Classifications:- 60H20, 45D05

Key Words - Fourier Inversion Formula, Gronwall Inequality, Ito-Volterra equation, Martingale, Positive Kernel, Stochastic differential equation

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SIGNIFICANCE AND EXPLANATION

A concept useful in establishing the stability of a solution to certain linear and non-linear integral equations is that of a positive kernel. This is a function $a(t)$ such that

$$(*) \qquad \int_0^T x(t) \int_0^t a(t-\tau)x(\tau)d\tau dt \geq 0$$

for every continuous function $x(t)$ and every $T \geq 0$. For example, if $\lambda \geq 0$ then $e^{-\lambda t}$ is a positive kernel. We show that this concept generalizes to a probabilistic setting, where stochastic integrals are used. But the analogous estimate (*) no longer holds. There is a significant correction term that must be added to the left side of (*). We provide this term, and consider its implication concerning energy estimates for certain stochastic integral equations. The one-dimensional linear homogeneous stochastic differential equation serves as an illustrative example.

POSITIVE KERNELS AND STOCHASTIC INTEGRALS

Marc A. Berger

INTRODUCTION

In the theory of Volterra equations of the form

$$x(t) + \int_0^t a(t-\tau)g(x(\tau))d\tau = f(t); \quad t \geq 0 \tag{V}$$

a concept useful for studying stability is that of a positive kernel $a(t)$. This is one for which

$$\int_0^T x(t) \int_0^t a(t-\tau)x(\tau)d\tau dt \geq 0$$

for every $x(t) \in C[0,\infty)$, and for every $T \geq 0$. A discussion of such kernels, and their implications concerning the stability of (V), appear in a large number of places, including MacCamy and Wong [4], and Nohel and Shea [5].

Let $a(t)$ be a positive kernel. Then if $F(t) \in C^1[0,\infty)$

$$\int_0^T x(t) \int_0^t a(t-\tau)x(\tau)dF(\tau)dF(t) \geq 0$$

for every $x(t) \in C[0,\infty)$, and for every $T \geq 0$. To extend the setting a bit, let $(\Omega,F,\mathbb{P})$ be a probability space. If $\{G(t) : t \geq 0\}$ is an a.s. differentiable process, then

$$\int_0^T \xi(t) \int_0^t a(t-\tau)\xi(\tau)dG(\tau)dG(t) \geq 0, \quad \text{a.s.} \tag{*}$$

for every a.s. continuous stochastic process $\{\xi(t) : t \geq 0\}$, and for every $T \geq 0$. Suppose now that there is a Brownian motion $\{\beta(t) : t \geq 0\}$ on $(\Omega,F,\mathbb{P})$. We consider the stochastic integral

$$I_a(T;\xi) = \int_0^T \xi(t) \int_0^t a(t-\tau)\xi(\tau)d\beta(\tau)d\beta(t); \quad T \geq 0$$

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where $\{\xi(t) : t \geq 0\}$ is a nonanticipating and a.s. continuous stochastic process. If it was possible to interpret the stochastic integral as a classical Stieltjes integral, then (*) would be valid with $G(t) = \beta(t)$. But of course, this is not so. In fact the estimate (*) no longer holds. For if $a(t) \equiv 1$ then

$$I_a(T;\xi) = \frac{1}{2}\left|\int_0^T \xi(t)d\beta(t)\right|^2 - \frac{1}{2}\int_0^T |\xi(t)|^2 dt, \quad \text{a.s.}; \quad T \geq 0 .$$

What is true, however, is that whenever $a(t)$ is a positive kernel

$$I_a(T;\xi) + \frac{1}{2}\,a(0)\int_0^T |\xi(t)|^2 dt \geq 0, \quad \text{a.s.}$$

for every $T \geq 0$. This is the content of the theorem in §1. The precise technical hypotheses on $a(t)$ and $\xi(t)$ are also provided in §1. In §2 we present an estimate concerning solutions of

$$\xi(t) + \int_0^t a(t-\tau)g(\xi(\tau))d\beta(\tau) = f(t); \quad t \geq 0$$

when they exist.

§1. Basic Estimate

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with a Brownian motion $\{\beta(t) : t \geq 0\}$. Let S be the space of real-valued functions $a(t) \in C^1[0,\infty)$ such that $e^{-\lambda t}a(t) \in L^1(0,\infty)$ for all $\lambda > 0$. A function $a(t) \in S$ is said to be of <u>positive type</u> if

$$(1.1) \qquad \int_0^T x(t) \int_0^t a(t-\tau)x(\tau)d\tau dt \geq 0$$

for every real-valued function $x(t) \in C[0,\infty)$, and for every $T \geq 0$. For a discussion and characterization of functions of positive type, and some illustrative examples, the reader is referred to MacCamy and Wong [4], and Nohel and Shea [5].

Theorem:

Let $a(t) \in S$ be of positive type. Then

$$(1.2) \qquad \int_0^T \xi(t) \int_0^t a(t-\tau)\xi(\tau)d\beta(\tau)d\beta(t) + \frac{1}{2} a(0) \int_0^T |\xi(t)|^2 dt \geq 0, \quad \text{a.s.}$$

for every real-valued stochastic process $\{\xi(t) : t \geq 0\}$ which has a stochastic differential $d\xi(t)$ with respect to $\beta(t)$, and for every $T \geq 0$.

The proof of this theorem relies on the following three results.

Lemma 1 (Correction Formula):

Let $\{\phi(\tau,t) : 0 \leq \tau \leq t \leq T\}$ be a real-valued t-nonanticipating stochastic process which has a stochastic differential $\partial_\tau \phi(\tau,t)$ with respect to $\beta(\tau)$, and satisfies

$$(1.3) \qquad \int_0^T \int_0^t |\phi(\tau,t)|^2 d\tau dt < \infty, \quad \text{a.s.}; \quad \int_0^T |\phi(t,t)| dt < \infty, \quad \text{a.s.}$$

Then

$$(1.4) \qquad \int_0^T \int_\tau^T \phi(\tau,t)d\beta(t)d\beta(\tau) = \int_0^T \int_0^t \phi(\tau,t)d\beta(\tau)d\beta(t) + \int_0^T \phi(t,t)dt, \quad \text{a.s.}$$

Lemma 2:

Let $\Pi = \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$. If $a(t) \in S$ is of positive type, then $\hat{a}(z) \geq 0$ for $z \in \Pi$, where

$$(1.5) \qquad \hat{a}(z) = \mathrm{Re} \int_0^\infty e^{-zt} a(t)dt .$$

Lemma 3:

If $a(t) \in L^1(-\infty,\infty)$ is an even function, then for a.e. $t \in (-\infty,\infty)$

$$(1.6) \qquad a(t) = \frac{1}{\pi} \int_{-\infty}^\infty e^{i\tau t} \hat{a}(i\tau)d\tau .$$

Lemma 1 can be found in Berger [1], or Berger and Mizel [3]. The definitions of t-nonanticipating, $\partial_\tau \phi(\tau,t)$ and the two double integrals in (1.4) are also provided there. Lemma 2 can be found in Nohel and Shea [5]. Lemma 3 can be proved directly from the Fourier Inversion Formula.

Proof of Theorem:

Following Nohel and Shea [5] extend $a(t)$ evenly to $(-\infty,\infty)$; that is, $a(t) = a(-t)$, $t < 0$. And define $a_\lambda(t) \in L^1(-\infty,\infty)$, for $\lambda > 0$, by $a_\lambda(t) = e^{-\lambda|t|} a(t), t \in (-\infty,\infty)$. Let

$$(1.7) \qquad \tilde{\xi}_T(t) = \int_0^T e^{-i\tau t} \xi(\tau)d\beta(\tau); \quad t \in (-\infty,\infty) .$$

By Lemma 1, for $T \geq 0$

$$(1.8) \qquad \int_0^T \xi(t) \int_0^t a(t-\tau)\xi(\tau)d\beta(\tau)d\beta(t) + \frac{1}{2} a(0) \int_0^T |\xi(t)|^2 dt = Q_a(T;\xi), \quad \text{a.s.}$$

where

$$(1.9) \qquad Q_a(T;\xi) \equiv \frac{1}{2} \int_0^T \xi(t) \int_0^T a(t-\tau)\xi(\tau)d\beta(\tau)d\beta(t); \quad T \geq 0 \cdot$$

Using the fact that (see Berger and Mizel [3])

$$(1.10) \qquad \lim_{\lambda \downarrow 0} \sup_{0 \leq t \leq T} [|a_\lambda(t) - a(t)| + |a_\lambda'(t) - a'(t)|] = 0; \quad T \geq 0$$

and Lemma 3, it follows that

$$Q_a(T;\xi) = \frac{1}{2} \lim_{\lambda \downarrow 0} \int_0^T \xi(t) \int_0^T a_\lambda(t-\tau)\xi(\tau)d\beta(\tau)d\beta(t), \quad \text{a.s.}$$

$$(1.11) \qquad = \frac{1}{2\pi} \lim_{\lambda \downarrow 0} \int_0^T \xi(t) \int_0^T [\int_{-\infty}^{\infty} e^{is(t-\tau)}\hat{a}(\lambda + is)ds]\xi(\tau)d\beta(\tau)d\beta(t), \quad \text{a.s.}$$

$$= \frac{1}{2\pi} \lim_{\lambda \downarrow 0} \int_{-\infty}^{\infty} |\tilde{\xi}_T(s)|^2 \hat{a}(\lambda + is)ds, \quad \text{a.s.}$$

Thus, by Lemma 2, $Q_a(T;\xi) \geq 0$, a.s. for every $T \geq 0$. And the result is apparent now from (1.8). ∎

We note that from the more general version of the Correction Formula which appears in Berger and Mizel [3], one obtains a generalization of (1.2) to continuous martingales $\{y(t) : t \geq 0\}$. Namely,

$$(1.12) \quad \int_0^T \xi(t) \int_0^t a(t-\tau)\xi(\tau)dy(\tau)dy(t) + \frac{1}{2} a(0) \int_0^T |\xi(t)|^2 d\langle y,y\rangle(t) \geq 0, \quad \text{a.s.}$$

§2. Ito-Volterra Equations

We consider next the Ito-Volterra equation

$$\text{(I-V)} \qquad \xi(t) + \int_0^t \sigma(t-\tau)g(\xi(\tau))d\beta(\tau) + \int_0^t b(t-\tau)\xi(\tau)d\tau = \varphi(t); \quad t \geq 0 .$$

In Berger [1] existence and uniqueness of a solution to (I-V) is established under the hypotheses

(a_1)
$$\sup_{0\leq t\leq T} |\sigma(t)| = \|\sigma\|_T < \infty, \quad \sup_{0\leq t\leq T} |b(t)| = \|b\|_T < \infty; \quad T \geq 0$$

(a_2) $g(x)$ is Lipschitz continuous on $\mathbb{R}$, and there exists a constant $K > 0$ such that for all $x \in \mathbb{R}$

$$\text{(2.1)} \qquad |g(x)|^2 \leq K(1 + |x|^2)$$

(a_3)
$$\sup_{0\leq t\leq T} \mathbb{E}|\varphi(t)|^2 < \infty; \quad T \geq 0$$

Let $b^*(t)$ be the resolvent kernel for $b(t)$. That is, $b^*(t)$ satisfies

$$\text{(2.2)} \qquad b^*(t) + \int_0^t b(t-\tau)b^*(\tau)d\tau = b(t); \quad t \geq 0 .$$

Then (I-V) can be put in the form

$$\text{(2.3)} \qquad \xi(t) + \int_0^t a(t-\tau)g(\xi(\tau))d\beta(\tau) = f(t); \quad t \geq 0$$

where

$$a(t) = \sigma(t) - \int_0^t b^*(t-\tau)\sigma(\tau)d\tau; \quad t \geq 0$$

$$\text{(2.4)} \qquad f(t) = \varphi(t) - \int_0^t b^*(t-\tau)\varphi(\tau)d\tau; \quad t \geq 0 .$$

For the linear case, $g(x) = x$, the solution $\xi(t)$ can be written in terms of a resolvent kernel, as is done in Berger [1], and Berger and Mizel [2]. The result is

$$\text{(2.5)} \qquad \xi(t) = f(t) + \int_0^t k(\tau,t)f(\tau)d\alpha(\tau), \quad \text{a.s.}; \quad t \geq 0$$

where

$$\alpha(t) = \beta(t) + a(0)t; \quad t \geq 0$$

(2.6) $$k(\tau,t) = \sum_{n=1}^{\infty} (-1)^n k_n(\tau,t); \quad 0 \leq \tau \leq t$$

$$k_1(\tau,t) = a(t-\tau), \quad k_{n+1}(\tau,t) = \int_{\tau}^{t} a(t-s)k_n(s,\tau)d\beta(\tau), \quad n = 1,2,\ldots; \quad 0 \leq \tau \leq t .$$

The definition of the integral on the right of (2.5) can be found in Berger [1]; or Berger and Mizel [2] or [3].

Corollary:

Let $a(t)$, given by (2.4), belong to S, and let $\xi(t)$ be a solution to (I-V). If $a(t)$ is of positive type, then

(2.7) $$\int_0^T g(\xi(t))[\xi(t) - f(t)]d\beta(t) \leq \frac{1}{2} a(0) \int_0^T |g(\xi(t))|^2 dt, \quad \text{a.s.}$$

for every $T \geq 0$, where $f(t)$ is given by (2.4). Similarly, if $-a(t)$ is of positive type, then

(2.8) $$\int_0^T g(\xi(t))[\xi(t)-f(t)]d\beta(t) \geq \frac{1}{2} a(0) \int_0^T |g(\xi(t))|^2 dt, \quad \text{a.s.}$$

for every $T \geq 0$..

Proof: Both (2.7) and (2.8) follow directly from (2.3) and (1.2). ■

As an example, consider the stochastic differential equation

$$d\xi(t) = -\mu\xi(t)d\beta(t) - \lambda\xi(t)dt; \quad t \geq 0$$

(2.9) $$\xi(0) = c .$$

This equation can be written in the form (2.3) with

(2.10) $$a(t) = \mu e^{-\lambda t}, \quad f(t) = ce^{-\lambda t}; \quad t \geq 0 .$$

If $\mu \geq 0$ and $\lambda \geq 0$ then $a(t)$ is of positive type, and the corollary provides the estimate

(2.11) $$\int_0^T \xi(t)[\xi(t) - f(t)]d\beta(t) \leq \frac{1}{2} \mu \int_0^T |\xi(t)|^2 dt, \quad \text{a.s.}$$

for every $T \geq 0$.

As another example, consider the pair of stochastic differential equations

$$d\xi(t) = -\mu\xi(t)d\beta(t) - \lambda\tilde{\xi}(t)dt; \quad t \geq 0$$

$$d\tilde{\xi}(t) = -\tilde{\mu}\tilde{\xi}(t)d\beta(t) - \tilde{\lambda}\tilde{\xi}(t)dt + \alpha\xi(t)dt; \quad t \geq 0$$

$$\xi(0) = c, \quad \tilde{\xi}(0) = \tilde{c}\ .$$

These equations can be combined and written in the form (2.3) where $a(t)$ and $f(t)$ are both solutions of the differential equation

$$x''(t) + \tilde{\lambda}x'(t) + \alpha\lambda x(t) = 0; \quad t \geq 0 \tag{2.13}$$

with initial conditions

$$a(0) = \mu, \quad a'(0) = -\lambda\tilde{\mu}; \quad f(0) = c, \quad f'(0) = -\lambda\tilde{c}\ . \tag{2.14}$$

If $\mu \geq 0$, $\tilde{\lambda} \geq 0$, $\tilde{\lambda}^2 \leq 4\alpha\lambda$, $\tilde{\lambda}\mu = 2\lambda\tilde{\mu}$ then $a(t)$ is of positive type, and the corollary provides the estimate (2.11) for every $T \geq 0$.

As a final example, consider the integro-differential equation (see Berger [1])

$$d\xi(t) = [\int_0^t \sigma(t-\tau)g(\xi(\tau))d\beta(\tau)]d\beta(t) + [\int_0^t b(t-\tau)g(\xi(\tau))d\tau]dt; \quad t \geq 0 \tag{2.15}$$

$$\xi(0) = c$$

If $\sigma(t)$ and $b(t)$ are both of positive type, then it follows from (1.1) and (1.2) that

$$G(\xi(T)) \geq G(c) - \sigma(0)\int_0^T |g(\xi(t))|^2 dt + \int_0^T g'(\xi(t))\left|\int_0^t \sigma(t-\tau)\xi(\tau)d\beta(\tau)\right|^2 dt, \quad \text{a.s.} \tag{2.16}$$

for every $T \geq 0$, where $G(x) = 2\int_0^x g(y)dy$, $x \in \mathbb{R}$.

If $g'(x) \geq 0$, $x \in \mathbb{R}$, then the last term on the right is positive, and can be dropped from the inequality. In fact, if $g(x) = x$ the Gronwall Inequality can be used to obtain the estimate

$$|\xi(T)| \geq |c|e^{-\frac{1}{2}\sigma(0)T}, \quad \text{a.s.} \tag{2.17}$$

for every $T \geq 0$. The reader can check that for the case $\sigma(t) \equiv \sigma \geq 0$, $b(t) \equiv 0$ the solution of (2.15) is a.s.

$$\xi(t) = ce^{-\frac{1}{2}\sigma t}\cosh\sqrt{\sigma}\beta(t), \quad t \geq 0. \tag{2.18}$$

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report is intended to expand the applicability of positive kernel theory to probabilistic settings and stochastic integrals. The main result states that if $a(t)$ is a positive kernel, and $\{\beta(t) : t \geq 0\}$ a Brownian motion, then

$$(*) \quad \int_0^T \xi(t) \int_0^t a(t-\tau)\xi(\tau)d\beta(\tau)d\beta(t) + \frac{1}{2}a(0)\int_0^T |\xi(t)|^2 dt \geq 0. \quad \text{a.s.}$$

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) for every stochastic process $\{\xi(t) : t \geq 0\}$ which has a stochastic differential $d\xi(t)$ with respect to $\beta(t)$, and for every $T \geq 0$. The implication of (*) concerning energy estimates for certain Ito-Volterra equations is discussed, and examples are provided.